PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A
MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Geomagnetic Westward Drift and Irregularities in the Earth's Rotation

M. G. Rochester

Phil. Trans. R. Soc. Lond. A 1960 **252**, 531-555
doi: 10.1098/rsta.1960.0014

GEOMAGNETIC WESTWARD DRIFT AND IRREGULARITIES IN THE EARTH'S ROTATION

By M. G. ROCHESTER

Department of Physics, University of Toronto, Toronto, Canada

(*Communicated by Sir Edward Bullard, F.R.S.—Received* 17 *August* 1959—*Revised* 4 *January* 1960)

CONTENTS

Bullard *et al.*'s (1950) rigid-sphere model for the steady westward drift of the earth's non-dipole magnetic field is extended to include the magnetic coupling provided by the presence of the higher multipole fields at the core-mantle boundary. If the first six harmonics of the observed surface field are taken into account, the retarding couple on the mantle is increased by a factor of 1·6. Using this result it would follow from Bullard's specialized model of the core that the geomagnetic dynamo mechanism generates a toroidal field of several hundred gauss in the deep interior of the core. Time-dependent perturbations of the mantle-core coupling are investigated rigorously, and it is shown that reasonable fluctuations of the fields at the core-mantle boundary are capable of explaining changes in the length of day at the rate of order 1 ms in 10 yr. The tightness of the coupling is increased by 60% over that afforded by Bullard's model. The argument provides additional evidence that the mean electrical conductivity in the bottom 2000 km of the mantle is at least 10^{-9} e.m.u. A summary of knowledge of the distribution of electrical conductivity with depth in the earth is given in the Introduction.

1. Introduction

Within the last 15 years a theory for the maintenance of the earth's magnetic field has been developed by Elsasser, Bullard and Parker. These attempts to construct a physically plausible model for the geomagnetic dynamo have been reviewed by Elsasser (1956).

For the purpose of the present paper we recall the first stage essential to the feedback mechanism proposed to regenerate the main dipole field. In the presence of convective exchange between the inner and outer parts of the fluid core, conservation of angular momentum requires that the deep interior of the core rotate more rapidly than the outer layers. This non-uniform rotation, together with the high electrical conductivity which requires that the magnetic field be 'transported' by the fluid, provides the mechanism by which a large toroidal magnetic field, with quadrupole symmetry about the rotational axis, is induced from the main dipole.

The presence of the perpendicular dipole component may be regarded as a transient phenomenon expressing some asymmetry in the convective régime of the core. The non-dipole part of the field exhibits the changes in time known as the secular variation, and is attributed to inductive interactions of the magnetic field and fluid motions in the outer layers of the core. The sources of the secular variation must lie within a few hundred kilometres below the core-mantle boundary, as shown by the restricted size of the centres of rapid change at the earth's surface, and the strong attenuation of hydromagnetic disturbances moving across substantial toroidal field under Coriolis acceleration (Tamao 1959). Lowes (1955) has argued that these regional field sources even contribute to the main dipole presently observed.

The waxing and waning of individual centres of change is accompanied by a general tendency of the centres of secular variation to move slowly westward over the surface of the earth (Bullard, Freedman, Gellman & Nixon 1950). The same tendency is exhibited by the non-dipole field contours. Lowes (1955) indicated that a large part of the entire secular variation was due to the westward drift of the regional field sources, and Whitham (1958) was able to assign at least half the secular variation to westward drift. Bullard and his collaborators determined the mean rate of drift, during 1905–45, of the non-dipole field as $0{\cdot}180 \pm 0{\cdot}015$ deg/yr, while during the same period the mean rate of drift of the isopors of the secular variation was found to be $0{\cdot}320 \pm 0{\cdot}067$ deg/yr. Whitham, on the other hand, found that the part of the same data appropriate to Canada yielded a westward drift of only one-third the worldwide average rate. The steady westward drift of individual tesseral harmonics differs for different harmonics, and in particular the perpendicular dipole appears not to have participated in the worldwide westward drift during the last century, its shift in that time not exceeding 5° (Bullard *et al.* 1950).

Bullard found no evidence for a worldwide drift in latitude, or for a dependence on latitude (apart from local fluctuations) of the rate of westward drift. Vestine's (1953) determination of a northward drift of the 'eccentric dipole', comparable in magnitude to its westward drift during the last century, is questionable because of the uncertainties in the data.

Because of the high conductivity of the fluid core, the non-dipole field lines are 'frozen' in the fluid and partake of its motions. Their movement westward with respect to an observer stationed on the mantle is evidence that the outer layers of the core do not rotate as rapidly as the mantle. This conclusion is supported by the hydromagnetic theory of boundary-layer phenomena at the top of the core, due to Elsasser & Takeuchi (1955). That the rotation of the mantle is not constrained to follow that of the outer core indicates the absence of appreciable mechanical coupling between the mantle and the core. Additional evidence for this is provided by the observations (Vestine 1953) that irregular fluctuations in the angular velocity of the mantle are accompanied by changes, in the same sense, of the rate of westward drift. A numerical calculation carried out by Bullard *et al.* (1950) demonstrates that viscous boundary-layer friction is unable to effect the coupling of the mantle to the core.

The generation by the geomagnetic dynamo of toroidal field in the core, together with the weak conductivity of the lower mantle which allows toroidal field to leak into the mantle, leads to an explanation of a steady westward drift maintained by electromagnetic forces on the mantle. Bullard *et al.* (1950) constructed a simple model for this electromagnetic coupling of the mantle to the core, in which the geomagnetic field was approximated by the axial dipole. In §3 of the present paper this model is extended to include the mechanical effects of the remainder of the earth's magnetic field. Perturbations of the electromagnetic coupling, and the consequent changes in the rate of rotation of the mantle, are discussed in §4.

1.1. *Electrical conductivity in the earth*

Observations made at the earth's surface can be used to infer the magnetic field effecting the coupling of the mantle to the core only if the electrical properties of the mantle and core are known. Within the last 10 years considerable progress has been made in quantitative estimates of the distribution of electrical conductivity with depth in the earth. The data now available have been set out in table 1. In the present paper we shall simplify the actual conductivity distribution by assuming the mantle to have uniform conductivity from the core boundary up for 2000 km, and to be insulating from there on to the top.

Table 1. Electrical conductivity in the earth

depth (km)	conductivity (e.m.u.)	reference	method of calculation
surface rocks	10^{-14} to 10^{-15}	Stratton (1941)	direct measurement
600 to 700	10^{-12}	Lahiri & Price (1939)	induction in conducting earth, accompanying transient ionospheric disturbances
800	10^{-11}		
900	10^{-10}	McDonald (1957)	amplitude attenuation and phase retardation of different harmonics of secular variation
1900	6×10^{-10}		
2900	2×10^{-9}		
900 to 2900	mean $\leqslant 10^{-9}$	Runcorn (1955)	rapidity of penetration through mantle of sudden changes in the rate of secular variation
900 to 2900	mean $\geqslant 10^{-9}$	present paper	tightness of electromagnetic mantle-core coupling required to explain rate of irregular changes in length of day
core (assumed uniform)	$1{\cdot}3\times10^{-5}$	Elsasser (1950)	extrapolation to core conditions of behaviour of iron at high temperatures and pressures
	3×10^{-6}	Bullard (1949*a*)	
	$> 10^{-8}$	Rikitake (1952)	absence of high intensity magneto-hydrodynamic waves in core
	$< 10^{-5}$	Rikitake (1952)	restriction on magnitude of perturbations causing the secular variation

1·2. *Index of notation*

In brackets following the symbol is the section of the paper in which the symbol is defined.

$\mathscr{A}_{nm}$ (§4·3*a*)

a radius of inner sphere in Bullard's core model

(a), (b) superscripts denoting toroidal fields generated by dynamo action inside core, and by induction at core boundary

B^{Pnm}_{TNm}	(§4·3*b*)
b	radius of core boundary, 3470 km
C^{Pnm}_{TNm}	(§3·1)
$\mathscr{C}_k$	(§4·4)
c	radius of upper boundary of conducting part of mantle, assumed 5470 km
d	earth's radius, 6370 km
$\mathbf{E}$	electric field
$\mathscr{E}(\lambda, p)$	(§§4·4, 4·5)
$\hat{e}_k$	unit vectors in spherical polar co-ordinates
F'_{nm}	(§3·1)
$F'^{(1)}_{nm}$	(§4·3)
f_{nm}	(§4·3*a*)
G_{Pnm}	(§4·3)
$\mathscr{G}_{Pnm}$	(§4·3*a*)
g^m_n, h^m_n	Schmidt coefficients in magnetic potential
$\mathbf{H}_{Pnm}$	poloidal magnetic field (§2)
$\mathbf{H}_{TNm}$	toroidal magnetic field (§3.2)
$\mathbf{H}^{Pnm}_{TNm}$	toroidal field induced from poloidal field at core boundary (§3·1)
$\mathscr{H}_{Pnm}, \mathscr{H}_{TNm}$	(§4·3)
I_c, I_m, I	moments of inertia of core, mantle, whole earth (§4·1)
$I_{nm, Nm}$	(§3·2)
$\mathscr{I}$	imaginary part of
$J_{n+\frac{1}{2}}, Y_{n+\frac{1}{2}}$	Bessel functions of first and second kind
$\mathbf{j}$	electric current density
K_N	(§3·1)
$\mathscr{K}_N$	(§4·3*b*)
$\hat{k}$	unit vector along axis of rotation
k	(§3·1)
k_1	(§4·3)
$\mathscr{L}_n$	(§4·4)
M	(§2)
$M^{(1)}$	(§4·3)
$\mathscr{M}$	(§4·3*c*)
m	azimuthal quantum number
n, N	zonal quantum number
$\mathscr{N}^N_n$	(§3·1).
P_{nm}	unnormalized associated Legendre coefficient
p	complex variable when time dependence removed by Fourier transform (§4·3)
$\bar{p}$	(§4·4)
$\mathscr{Q}_{nm}$	(§4·3*a*)
$\mathscr{R}$	real part of
r, θ, ϕ	spherical polar co-ordinates
$S, S_k, S^{(c)}_k$	sums of terms quadratic in Schmidt coefficients (§§3·2, 4·4)

S_c	surface of core boundary
dS	surface element
T	length of day
t	time
U	magnetic potential (§2)
V_m	volume of conducting part of mantle
dV	volume element
$\mathbf{v}$	velocity of conductor past reference frame
W	(§4·4)
x, y	(§4·5)
$Z, \mathfrak{Z}$	Bessel functions (§3·2)
α	(§4·3*a*)
Γ	total electromagnetic couple on mantle (§2)
$\Gamma_{\text{acc.}}$ $\Gamma_{\text{ret.}}$	accelerating, retarding couples on mantle (§3·2)
$\Gamma_{Pnm;\ Pnm}$	couple due to self-interaction of poloidal field with own current system (§3·2)
$\Gamma_{Pnm;\ {Pnm \atop TNm}}$	couple due to mutual interaction of poloidal and toroidal fields and currents (§3·2)
Δ	accumulated change in
δ_{mn}	Kronecker delta
λ	variable time parameter (§4·3*a*)
λ_n^m	quasi-normalizing coefficient in U (§2)
σ, σ_1	electrical conductivity of core, lower mantle
τ	time-constant of mantle-core coupling (§§4·1, 4·4)
τ_B	time-constant on Bullard model (§4·4)
$\hat{\phi}$	unit vector in east azimuth
χ_0	rate of drift of core past mantle (§3·1)
$\mathbf{X}_{Pnm}, \mathbf{X}_{TNm}$	(§4·3)
Ω	(§4·3)
ω_2, ω_m	eastward angular velocity of upper core, mantle
(0), (1)	superscripts denoting steady state and first-order perturbations therefrom.

Complex conjugates are denoted by asterisks and are abbreviated to c.c.

2. Bullard's model for the steady westward drift

In this model (figure 1) the mantle, rotating with uniform angular velocity ω_m, is electromagnetically coupled to the core by two opposing forces, due to the penetration into the conducting region of the mantle of:

(*a*) a toroidal field diffusing out from the deep interior of the core, where it is induced as part of the dynamo mechanism, and

(*b*) another toroidal field induced from the main dipole field by the differential angular velocity at the core boundary. The outer layers of the core are approximated by a rigid spherical shell rotating with angular velocity $\omega_2 < \omega_m$. We need not specify the quantitative details of the generation of toroidal field of type (*a*); it is enough for us that such a field, of quadrupole symmetry and of the sense compatible with the non-uniform rotation of the core, reach and penetrate the core boundary.

The ponderomotive ($\mathbf{j} \times \mathbf{H}$) interaction of the current producing the toroidal field (a) with the field of the main dipole provides a couple tending to accelerate the mantle. Toroidal field (b), being in the opposite sense, interacts with the main dipole so as to retard the mantle. Lenz's law leads to a simple qualitative explanation for the sense of the couples produced; the presence of toroidal field (a) in the mantle tends to force it to rotate with the interior of the core, while toroidal field (b) tends to force it to rotate in coincidence with the outer layers of the core. Clearly a steady-state equilibrium is possible when the two couples cancel one another, and the mantle follows a sort of weighted mean motion of the core. In Bullard's model, at equilibrium the toroidal fields (a) and (b) annul one another throughout the mantle. This is no longer true when the extra coupling provided by the non-dipole part of the earth's field is taken into account, although for a stationary rotation of the mantle the net magnetic couple exerted on it must vanish. The parts of Bullard's analysis essential to the present paper are reproduced here with a slightly different notation.

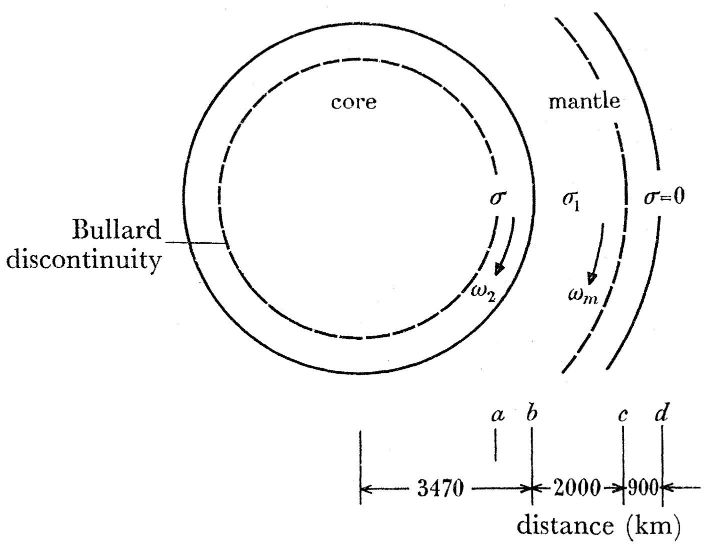

FIGURE 1. Rigid-sphere earth model.

The earth's magnetic field of internal origin, as observed at the surface of the Earth ($r = d$), can be represented as

$$\mathbf{H} = \sum_{n \geqslant 1} \sum_{m=0}^{n} \mathbf{H}_{Pnm} = \operatorname{grad} U, \tag{1}$$

where the subscript P designates a field of poloidal type and where

$$U = d \sum_{n \geqslant 1} \sum_{m=0}^{n} \left(\frac{d}{r}\right)^{n+1} [g_n^m \cos m\phi + h_n^m \sin m\phi] \, \lambda_n^m P_{nm}. \tag{2}$$

Here g_n^m and h_n^m are the Schmidt coefficients traditional to terrestrial magnetism, $P_{nm}(\cos\theta)$ is the unnormalized associated Legendre coefficient described by Stratton (1941, pp. 401–2), and λ_n^m is a quasi-normalizing factor such that

$$(\lambda_n^m)^2 = (2 - \delta_{m0}) \frac{(n-m)!}{(n+m)!}.$$

In Bullard's model the earth's field is replaced by the axial dipole $\mathbf{H}_{P10}$ which penetrates the mantle without screening: its radial component at the core-mantle boundary (in our notation $r = b$) is

$$-2g_1^0(d/b)^3 \cos\theta.$$

In the rigid-sphere model the production of toroidal field (b) occurs only at the spherical surface across which there is a discontinuity in angular velocity, and so appears via the boundary conditions. The poloidal dipole and toroidal quadrupole magnetic fields are independently continuous across $r = b$, but are related by the requirement of continuity in the tangential component of the electric field

$$\mathbf{E} = \frac{1}{4\pi\sigma}\,\text{curl}\,\mathbf{H} - \mathbf{v}\times\mathbf{H} \quad (\mathbf{v} = \omega r \sin\theta\hat{\phi}). \tag{3}$$

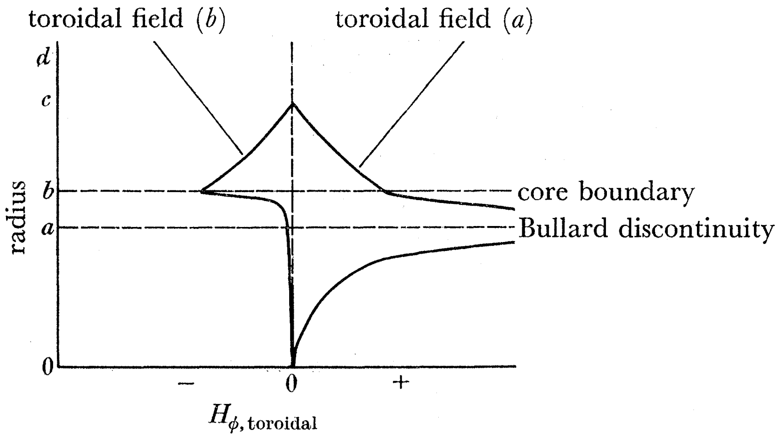

FIGURE 2. Steady-state toroidal fields (Bullard).

This boundary condition reduces to the equation

$$\frac{\partial}{\partial r}[rH_{\phi,\,T20}]_{r=b+} = \frac{\sigma_1}{\sigma}\frac{\partial}{\partial r}[rH_{\phi,\,T20}]_{r=b-} - 4\pi\sigma_1(\omega_m-\omega_2)\,b^2\sin\theta H_{r,\,P10}|_{r=b},$$

where

$$\mathbf{H}_{T20} = \mathbf{H}_{T20}^{(a)} + \mathbf{H}_{T20}^{(b)}$$

is the net toroidal field leaking into the mantle, and σ_1, σ are the (assumed uniform) electrical conductivities of the lower mantle and core, respectively. In view of table 1 we can neglect σ_1/σ with respect to 1. The toroidal fields are subject to one more condition, as a result of their inability to penetrate insulators: they must vanish at the top of the conducting region of the mantle, $r = c$.

The assumption of discontinuities in conductivity and angular velocity at $r = b$ simply introduces a discontinuity in the radial component of the gradient of the (entirely azimuthal) toroidal fields (figure 2). If we define the maximum value M attained by the toroidal field (a) at the core boundary, then in the mantle

$$\mathbf{H}_{\phi,\,T20}^{(a)} = -\tfrac{2}{3}M\left(\frac{b}{r}\right)^3 \frac{(1-r^5/c^5)}{(1-b^5/c^5)}\,\frac{\mathrm{d}P_{20}}{\mathrm{d}\theta}. \tag{4}$$

The toroidal field (b) in the mantle is

$$H_{\phi,\,T20}^{(b)} = \tfrac{4}{3}\pi\sigma_1(\omega_m-\omega_2)\,b^2g_1^0\left(\frac{d}{r}\right)^3 \frac{(1-r^5/c^5)}{(1+\frac{3}{2}b^5/c^5)}\,\frac{\mathrm{d}P_{20}}{\mathrm{d}\theta}.$$

The net couple on the mantle, in the sense tending to accelerate its rotation, is

$$\Gamma = \hat{k}\cdot\int_{V_m} \mathbf{r}\times(\mathbf{j}\times\mathbf{H})\,\mathrm{d}V = \frac{1}{4\pi}\int_{V_m} r\sin\theta\,[\mathrm{curl}\,\mathbf{H}\times\mathbf{H}]_\phi\,\mathrm{d}V, \tag{5}$$

where $\hat{k}$ is a unit vector along the axis of rotation, and the integration is over the conducting volume of the mantle. This reduces to

$$\Gamma = -\frac{1}{4\pi}\int_{S_c} r\sin\theta\,H_{r,\,P10}\,H_{\phi,\,T20}\,\mathrm{d}S$$

integrated over the core boundary. Thus, in the Bullard model, the couple vanishes when

$$H_{\phi,\,T20}\,|_{r=b} = 0.$$

The steady rate of westward drift is

$$\omega_m - \omega_2 = \frac{M}{2\pi\sigma_1 b^2 g_1^0}\left(\frac{b}{d}\right)^3 \frac{(1+\frac{3}{2}b^5/c^5)}{(1-b^5/c^5)}. \tag{6}$$

So long as the geomagnetic dynamo is maintained, stationarity is possible only when there is a steady westward drift. Lenz's law can be invoked to establish the stability of this equilibrium, for any disturbance from equilibrium brings about the action of a restoring couple due to the predominance in the mantle of one of the toroidal fields (*a*), (*b*) over the other.

3. Retarding effects of the non-dipole fields

3·1. *Non-dipole fields in the mantle*

If one extrapolates the non-dipole field observed at the surface of the earth down to the core boundary, one finds that the magnetic topography there is very rough indeed, even if one assumes an insulating mantle (McDonald 1955). Because of the factors $(d/r)^{n+2}$ in the expressions for the higher harmonics in field strength, the non-dipole fields increase rapidly with depth and at the core are nearly as strong as the dipole. In Bullard's model for the westward drift the non-dipole field played a passive role, simply being swept around with the outer layers of the core and taking no part in coupling the mantle to the core. Clearly, however, as the non-dipole fields sweep across the core-mantle boundary, corresponding toroidal fields are induced there and diffuse out through the mantle. The $\mathbf{j}\times\mathbf{H}$ interactions of the induced currents producing these toroidal fields, with their primaries (the poloidal harmonics), bring about an additional couple which retards the mantle. There is also a small retarding couple provided by the tendency of the mantle to overcome the screening of tesseral multipole poloidal fields as they move (attached to the outer core) westward relative to the mantle.

To make the problem tractable we assume:

(1) that the multipole components of the earth's field do not change in strength, so that for each n, m

$$(g_n^m)^2 + (h_n^m)^2 = \text{constant};$$

(2) that all the higher harmonics, including the perpendicular dipole, have the same steady westward drift relative to the mantle. If we set

$$\chi_0 = \omega_2 - \omega_m$$

then χ_0 is negative for the observed drift.

We require the poloidal and toroidal solutions of the induction equation

$$\frac{\partial \mathbf{H}}{\partial t} = -\frac{1}{4\pi\sigma_1}\operatorname{curl}\operatorname{curl}\mathbf{H} - \omega_m \sum_{k=1}^{3} \hat{e}_k \frac{\partial H_k}{\partial \phi} \tag{7}$$

for uniformly rotating fields in a steadily rotating spherically symmetric region of uniform conductivity. (Here the $\hat{e}_k$ are unit vectors for spherical polar co-ordinates.) The necessary mathematical techniques have been outlined by Bullard (1949*b*). The last term on the right-hand side of (7) vanishes if we take the frame of reference as rotating with uniform angular velocity ω_m. It is convenient to work not with the actual fields but with their complex representations, of which the actual fields are the real parts.

We define a semi-imaginary parameter k such that

$$k^2 = 4\pi\sigma_1 \mathrm{i}m\chi_0.$$

Since the tesseral fields have a time factor $\mathrm{e}^{-\mathrm{i}m\chi_0 t}$, (7) reduces to

$$\operatorname{curl}\operatorname{curl}\mathbf{H} = k^2\mathbf{H}. \tag{8}$$

Then the poloidal field with quantum numbers n, m has components

$$H_{r,Pnm} = -(n+1)\,F'_{nm}(b/c)^{n+2}\,\tfrac{1}{2}\pi(kc)^{\frac{3}{2}}\,Y_{n-\frac{1}{2}}(kc)\,\mathcal{N}_n^{n-1}(kr)\,P_{nm}\,\mathrm{e}^{\mathrm{i}m(\phi-\chi_0 t)}$$
$$\sim -(n+1)\,F'_{nm}(b/r)^{n+2}P_{nm}\,\mathrm{e}^{\mathrm{i}m(\phi-\chi_0 t)}, \tag{9}$$

$$H_{\phi,Pnm} = \mathrm{i}mF'_{nm}(b/c)^{n+2}\,\tfrac{1}{2}\pi(kc)^{\frac{3}{2}}\,Y_{n-\frac{1}{2}}(kc)\left[\mathcal{N}_n^{n-1}(kr) - \frac{kr}{n}\mathcal{N}_{n-1}^{n-1}(kr)\right]\frac{P_{nm}}{\sin\theta}\,\mathrm{e}^{\mathrm{i}m(\phi-\chi_0 t)}$$
$$\sim \mathrm{i}mF'_{nm}\left(\frac{b}{r}\right)^{n+2}\frac{P_{nm}}{\sin\theta}\,\mathrm{e}^{\mathrm{i}m(\phi-\chi_0 t)}\left[1-\frac{(kr)^2}{n(2n-1)}\left\{1-\left(\frac{r}{c}\right)^{2n-1}\right\}\right] \tag{10}$$

in the conducting part of the mantle. $H_{\theta,Pnm}$ is not written down because we shall not need it. Here

$$\mathcal{N}_n^N(kr) = (kr)^{-\frac{3}{2}}\left[J_{n+\frac{1}{2}}(kr) - \frac{J_{N+\frac{1}{2}}(kc)}{Y_{N+\frac{1}{2}}(kc)}\,Y_{n+\frac{1}{2}}(kr)\right],$$

where $J_{n+\frac{1}{2}}(x)$ and $Y_{n+\frac{1}{2}}(x)$ are Bessel functions of the first and second kind and the complex parameter F'_{nm} is defined in terms of the Schmidt coefficients at epoch t by extrapolating down to $r=c$:

$$\mathscr{R}(F'_{nm}\,\mathrm{e}^{-\mathrm{i}m\chi_0 t}) = g_n^m\lambda_n^m(d/b)^{n+2},$$
$$\mathscr{I}(F'_{nm}\,\mathrm{e}^{-\mathrm{i}m\chi_0 t}) = -h_n^m\lambda_n^m(d/b)^{n+2}.$$

The approximations in (9) and (10) are obtained from the series representations for Bessel functions of small argument and are good to within 10% for $\sigma_1 \sim 10^{-9}$ e.m.u. and $|\chi_0| \sim 10^{-10}\,\mathrm{s}^{-1}$. This indicates that the steady-state poloidal fields (particularly the radial component) are not much shielded by the conducting lower mantle.

Applying the boundary condition (3) we find that $\mathbf{H}_{Pnm}$ induces at $r=b$ toroidal fields of the same tesseral character, but with one more and one less zonal node, namely $\mathbf{H}^{Pnm}_{T,n+1,m}$ and $\mathbf{H}^{Pnm}_{T,n-1,m}$, respectively. When $n=1$ or $m=n$ no $\mathbf{H}^{Pnm}_{T,n-1,m}$ field is produced. Since σ_1/σ is small, the core acts as a nearly perfect reflector of toroidal fields produced at its boundary, and the diffusion into the mantle of these fields is unaffected by the state of motion deep in the core. On neglecting σ_1/σ with respect to 1, the boundary condition reduces to

(i) $m=0$

$$\sum_N \frac{\partial}{\partial r}[rH_{\phi,\,TN0}^{Pn0}]_{r=b+} = 4\pi\sigma_1\chi_0 b^2 \sin\theta\, H_{r,Pn0}|_{r=b},$$

(ii) $m \neq 0$

$$\sum_N N(N+1)\frac{\partial}{\partial r}[rH_{\theta,\,TNm}^{Pnm}]_{r=b+} = \frac{(kb)^2}{\sin^2\theta}\frac{\partial}{\partial\theta}[\sin^2\theta\, H_{r,Pnm}]_{r=b},$$

where $N = n \pm 1$. With the aid of recurrence relations for the associated Legendre coefficients and the condition that no toroidal field penetrate beyond $r = c$, we find that

$$H_{\theta,\,TNm}^{\;Pnm} = \mathrm{i}mC_{TNm}^{Pnm}kr\mathcal{N}_N^N(kr)\,(P_{Nm}/\sin\theta)\,\mathrm{e}^{\mathrm{i}m(\phi-\chi_0 t)}, \tag{11}$$

$$H_{\phi,\,TNm}^{\;Pnm} = -C_{TNm}^{Pnm}kr\mathcal{N}_N^N(kr)\,(\mathrm{d}P_{Nm}/\mathrm{d}\theta)\,\mathrm{e}^{\mathrm{i}m(\phi-\chi_0 t)}, \tag{12}$$

where
$$C_{T,n+1,m}^{Pnm}kb\mathcal{N}_{n+1}^{n+1}(kb)\,[(n+1)-K_{n+1}] = 4\pi\sigma_1\chi_0 b^2F'_{nm}\frac{(n-m+1)}{(2n+1)}, \tag{13}$$

$$C_{T,n-1,m}^{Pnm}kb\mathcal{N}_{n-1}^{n-1}(kb)\,[(n-1)-K_{n-1}] = -4\pi\sigma_1\chi_0 b^2F'_{nm}\frac{(n+1)\,(n+m)}{n(2n+1)}. \tag{14}$$

Here
$$K_N = \frac{kb\mathcal{N}_{N-1}^N(kb)}{\mathcal{N}_N^N(kb)}.$$

The physical significance of K_N is seen from the relation

$$\left.\frac{b\,\partial H_{\phi,TNm}^{Pnm}/\partial r}{H_{\phi,\,TNm}^{\;Pnm}}\right|_{r=b+} = K_N-(N+1).$$

In the limit $m = 0$,
$$K_N = \frac{2N+1}{1-(c/b)^{2N+1}}. \tag{15}$$

Hence, as N increases, the role of the boundary condition at $r = c$ in causing $\mathbf{H}_{TNm}^{Pnm}$ to diminish in strength with distance up from the core boundary, becomes less important relative to the simple multipole-type decrease in field strength, so far as points near the bottom of the mantle are concerned. The situation is aggravated by an increase in m, which introduces screening and renders the boundary condition at $r = c$ even less important at the bottom of the mantle. A numerical investigation shows that for

$$m = 0,\quad N > 3;\qquad m \neq 0,\quad N > 1$$

we can neglect $|K_N|$ entirely with respect to N. $K_N|_{m=0}$ is determined from (15) and, with the values of k, b and c appropriate to the model, we find $K_1|_{m=1} \sim -1$. The errors introduced by these approximations are less than 5 %.

Even though $k = 0$ when $m = 0$,

$$\lim_{m=0} C_{TNm}^{Pnm}kr\mathcal{N}_N^N(kr) = [C_{TNm}^{Pnm}kb\mathcal{N}_N^N(kb)]_{m=0}\left(\frac{b}{r}\right)^{N+1}\frac{(1-\{r/c\}^{2N+1})}{(1-\{b/c\}^{2N+1})},$$

where the first factor is given by (13) or (14).

3·2. *Electromagnetic couples on the mantle*

It is readily shown that a net couple on the mantle, about the rotational axis, is produced by the Lenz-law interaction of:

(1) each field $\mathbf{H}_{Pnm}$ with the current system $\mathbf{j} = (4\pi)^{-1}\,\mathrm{curl}\,\mathbf{H}$ of each of the following fields;

(2) each $\mathbf{j} = (4\pi)^{-1}\,\mathrm{curl}\,\mathbf{H}_{Pnm}$ with each of the following fields:

(i) $\mathbf{H}_{T,n-1,m}^{P,n-2,m}$ $(n \geq 3,\ 0 \leq m \leq n-2)$,
(ii) $\mathbf{H}_{T,n-1,m}^{Pnm}$ $(n \geq 2,\ 0 \leq m \leq n-1)$,
(iii) $\mathbf{H}_{T,n+1,m}^{Pnm}$ $(n \geq 1,\ 0 \leq m \leq n)$,
(iv) $\mathbf{H}_{T,n+1,m}^{P,n+2,m}$ $(n \geq 1,\ 0 \leq m \leq n)$,
(v) $\mathbf{H}_{Pnm}$ $(n \geq 1,\ 1 \leq m \leq n)$.

In calculating the couples we must use the real fields, i.e. half the sum of the complex field and its conjugate. For each of the interactions (i) to (v) above we need the following results:

(*a*) a theorem on the integration of products of Bessel functions $Z_p(\mu x)$, $\mathfrak{Z}_p(\nu x)$ (Watson 1944, p. 134):

$$(\mu^2-\nu^2)\int_{x_1}^{x_2} xZ_p(\mu x)\,\mathfrak{Z}_p(\nu x)\,\mathrm{d}x = [\nu x Z_p(\mu x)\,\mathfrak{Z}_{p-1}(\nu x) - \mu x Z_{p-1}(\mu x)\,\mathfrak{Z}_p(\nu x)]_{x_1}^{x_2}. \tag{16}$$

This is applicable when $m \neq 0$ (the axisymmetric integrations present no difficulty);

(*b*) the recurrence relation

$$\frac{2p}{\mu x} Z_p(\mu x) = Z_{p-1}(\mu x) + Z_{p+1}(\mu x)$$

and for (i) to (iv) the result (readily obtained from the expressions given by Stratton, p. 402)

$$I_{nm,\,Nm} = \int_0^\pi P_{nm}\frac{\mathrm{d}P_{Nm}}{\mathrm{d}\theta}\sin^2\theta\,\mathrm{d}\theta$$

$$= \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}\left[\frac{(n-1)\,(n-m)}{(2n-1)}\,\delta_{n,\,N+1} - \frac{(n+2)\,(n+m+1)}{(2n+3)}\,\delta_{n,\,N-1}\right]. \tag{17}$$

and for (v) the result
$$\int_0^\pi (P_{nm})^2\sin\theta\,\mathrm{d}\theta = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}. \tag{18}$$

Applying (16) we find that the expression for the couple (5) reduces in each case to an integral over the core boundary, namely (i) to (v) become

$$\Gamma_{Pnm;\,{}^{P,\,n-2,\,m}_{T,\,n-1,\,m}} = -\frac{1}{4\pi}\int_{S_c} r\sin\theta\,\mathrm{d}S\,\mathscr{R}(H_{r,\,Pnm})\,\mathscr{R}(H_{\phi,\,{}^{P,\,n-2,\,m}_{T,\,n-1,\,m}})$$

and three analogous expressions, and

$$\Gamma_{Pnm;Pnm} = -\frac{1}{4\pi}\int_{S_c} r\sin\theta\,\mathrm{d}S\,\mathscr{R}(H_{r,\,Pnm})\,\mathscr{R}(H_{\phi,\,Pnm}),$$

respectively. The total couple on the mantle, due to the westward drift of the outer core, is therefore

$$\Gamma_{\text{ret.}} = \sum_{n\geqslant 3}\sum_{m=0}^{n-2}\Gamma_{Pnm;\,{}^{P,\,n-2,\,m}_{T,\,n-1,\,m}} + \sum_{n\geqslant 2}\sum_{m=0}^{n-1}\Gamma_{Pnm;\,{}^{Pnm}_{T,\,n-1,\,m}} + \sum_{n\geqslant 1}\sum_{m=0}^{n}\Gamma_{Pnm;\,{}^{Pnm}_{T,\,n+1,\,m}}$$

$$+\sum_{n\geqslant 1}\sum_{m=0}^{n}\Gamma_{Pnm;\,{}^{P,\,n+2,\,m}_{T,\,n+1,\,m}} + \sum_{n\geqslant 1}\sum_{m=1}^{n}\Gamma_{Pnm;Pnm}. \tag{19}$$

In the steady state this retarding couple is balanced by the couple

$$\Gamma_{\text{acc.}} = -\frac{1}{4\pi}\int_{S_c} r\sin\theta\,\mathrm{d}S\,[H_{r,\,P10}+H_{r,\,P30}]\,H_{\phi,\,T20}^{(a)} \tag{20}$$

due to the penetration of the mantle by the toroidal field which is generated by the dynamo processes.

Since $\mathbf{j}\times\mathbf{H}$ interactions other than those set out in (i) to (v) above do not contribute a couple on the mantle, and $H_{r,\,Tnm} = 0$, we may write the total couple on the mantle (the sum of (19) and (20)) as

$$\Gamma = -\frac{1}{4\pi}\int_{S_c} r\sin\theta\,\mathrm{d}S\left[\sum_{n\geqslant 1}\sum_{m=0}^{n}(H_{r,\,Pnm}+H_{r,\,Tnm})\right]\left[\sum_{N\geqslant 1}\sum_{M=0}^{N}(H_{\phi,\,PNM}+H_{\phi,\,TNM})\right],$$

where the real parts of the fields are to be understood, and

$$\mathbf{H}_{Tnm} = \mathbf{H}^{(a)}_{T20}\delta_{nm,\,20} + \mathbf{H}^{P,\,n-1,\,m}_{Tnm} + \mathbf{H}^{P,\,n+1,\,m}_{Tnm}.$$

It follows that in the steady state

$$\int_{V_m} r\sin\theta\,[\mathbf{H}\times\operatorname{curl}\mathbf{H}]_\phi\,\mathrm{d}V = \int_{S_c} r\sin\theta\,H_r H_\phi\,\mathrm{d}S. \tag{21}$$

Though it seems likely that a judicious application of a divergence theorem and the boundary conditions at $r = c$ should transform the volume integral to the surface integral, we have not been able to achieve this, and have had to proceed through the interactions of each harmonic one by one.

For purposes of calculation, however, the expressions (19) and (20) are more suitable. We write

$$\Gamma_{\text{ret.}} = \pi\sigma_1\chi_0 b^5 \sum_{k=1}^{5} S_k,$$

where the sums S_k are obtained from (19) in the order there written, as follows

$$S_1 = S_4 = -4\sum_{n\geqslant 1}\sum_{m=0}^{n}\{g^m_n g^m_{n+2} + h^m_n h^m_{n+2}\}\frac{(n+3)\,(n+m+1)\,(n+m+2)}{(2n+1)\,(2n+3)\,(2n+5)}\,\frac{\lambda^m_{n+2}}{\lambda^m_n}\left(\frac{d}{b}\right)^{2(n+3)},$$

$$S_2 = 4\sum_{n\geqslant 2}\sum_{m=0}^{n-1}\{(g^m_n)^2 + (h^m_n)^2\}\frac{(n+1)^2\,(n^2-m^2)}{n(2n-1)\,(2n+1)^2}\left(\frac{d}{b}\right)^{2(n+2)},$$

$$S_3 = 4\sum_{n\geqslant 1}\sum_{m=0}^{n}\{(g^m_n)^2 + (h^m_n)^2\}\frac{(n+2)\,[(n+1)^2-m^2]}{(2n+1)^2\,(2n+3)}\left(\frac{d}{b}\right)^{2(n+2)},$$

$$S_5 = 4\sum_{n\geqslant 1}\sum_{m=1}^{n}\{(g^m_n)^2 + (h^m_n)^2\}\frac{m^2(n+1)}{n(4n^2-1)}\left\{1-\left(\frac{b}{c}\right)^{2n-1}\right\}\left(\frac{d}{b}\right)^{2(n+2)}.$$

Here we have ignored K_N. This is corrected for by multiplying the terms $S_{k,\,nm}$ as follows:

(*a*) $S_{2,\,20}$ and $S_{2,\,21}$ by 0·50;

(*b*) $S_{4,\,10}$, $S_{2,\,30}$ and $S_{3,\,10}$ by 0·78;

(*c*) $S_{4,\,20}$, $S_{2,\,40}$ and $S_{3,\,20}$ by 0·90.

The Schmidt coefficients for epoch 1955 are tabulated by Finch & Leaton (1957). The sums corrected for K_N are designated by a superscript (c). If they are cut off at $n = 6$ we find

$$S^{(c)}_1 = S^{(c)}_4 = 0{\cdot}29\ \text{e.m.u.},\qquad S^{(c)}_2 = 0{\cdot}58\ \text{e.m.u.},$$

$$S^{(c)}_3 = 3{\cdot}75\ \text{e.m.u.},\qquad S_5 = 0{\cdot}49\ \text{e.m.u.},$$

so that

$$S = \sum_{k=1}^{4} S^{(c)}_k + S_5 = 5{\cdot}40\ \text{e.m.u.}$$

Setting

$$\Gamma_{\text{acc.}} + \Gamma_{\text{ret.}} = 0,$$

we get, using (19) and (20),

$$-\pi\sigma_1\chi_0 b^5 S = \tfrac{8}{15} g^0_1 M d^3\left[1 - \frac{6}{7}\frac{g^0_3}{g^0_1}\left(\frac{d}{b}\right)^2\right]. \tag{22}$$

This expression gives a value of M larger than that obtained in Bullard's model (6) by a factor

$$\tfrac{15}{16} S (g^0_1)^{-2}\left(\frac{b}{d}\right)^6 \frac{(1+\frac{3}{2}b^5/c^5)}{(1-b^5/c^5)}\left[1-\frac{6}{7}\frac{g^0_3}{g^0_1}\left(\frac{d}{b}\right)^2\right]^{-1} \sim 1{\cdot}6.$$

If we set $\omega_m - \omega_2 = 0{\cdot}180$ deg/yr $= 10^{-10}\,\mathrm{s}^{-1}$ and take $\sigma_1 \sim 10^{-9}$ e.m.u., we find

$$M \sim 0{\cdot}2\,\mathrm{G}. \tag{23}$$

Bullard specialized the non-uniform rotation of the core's interior by a discontinuity in angular velocity at $r = a$ (our notation reverses the a, b of Bullard's paper). Reference to his paper (1950, p. 89) shows that (23) corresponds to a maximum toroidal field (a) in the core, on the surface $r = a$, of strength

$$\frac{2}{5}\frac{\sigma}{\sigma_1}M\left(\frac{b}{a}\right)^3\left(1-\frac{a^5}{b^5}\right) \sim 300\,\mathrm{G} \tag{24}$$

(taking $\sigma \sim 3\times10^{-6}$ e.m.u. and $a/b \sim 0{\cdot}80$). Such a large toroidal field in the core is not incompatible with the stability of the geomagnetic dynamo, as shown by Rikitake (1956). Reversing the argument, we can submit the existence of the westward drift as evidence for a very large toroidal field in the core. It should be pointed out here, however, that the argument from (23) to (24) is probably grossly oversimplified, in view of the complicated pattern of turbulence in the core, governed by Coriolis and magnetic forces.

Either side of (22) certainly sets a rough upper limit on the order of magnitude of the unbalanced electromagnetic couple available to act on the mantle

$$-\pi\sigma_1\chi_0 b^5 S \sim 0{\cdot}86\times10^{25}\,\mathrm{dyn\,cm}.$$

Elsasser & Munk (1958) have shown that if such a couple acted about an axis inclined to the rotational axis, it would be inadequate by a factor of 10^5 to effect the exchange of angular momentum between core and mantle required to account for the observed pole migration in the last 50 years.

4. Perturbations of the mantle-core coupling

4·1. *Changing core motions and irregularities in the earth's rotation*

After the secular deceleration of the earth's rate of rotation due to tidal friction, and the seasonal variations in ω_m due to atmospheric changes, are removed, there remains a further fault in the earth's time-keeping. The length of the day is subject to irregular fluctuations, once thought to be intermittent and catastrophic in nature, but shown by Brouwer (1952) to be capable of representation as a series of random cumulative changes. The rate of change of ω_m is such as to lengthen or shorten the day by a few milliseconds in a decade. Munk & Revelle (1952) concluded that changes in the motions in the fluid core offered the only reasonable geophysical possibility for causing these irregularities. They suggested that the electromagnetic coupling of the mantle to the core, invoked by Bullard to explain the westward drift, could effect the requisite transfer of angular momentum between the core and the mantle. Further weight was given to this suggestion by Vestine (1953), who matched the changes in the rate of westward drift over a 60 yr period with an inverse pulse in the length of the day, observed during the same time.

An appeal to the conservation of angular momentum of the core-mantle system provides a quantitative demonstration that a small change in the observed rate of westward drift is compatible with the observed changes in the length of the day. Suppose that the entire fluid core moves as a rigid body during the change (most of the core's rotational inertia is in the outer layers), then

$$I_m\Delta\omega_m + I_c\Delta\omega_2 = 0,$$

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

where the moments of inertia of the core and mantle are, respectively (Elsasser & Takeuchi 1955)

$$I_c = 8{\cdot}5 \times 10^{43}\,\mathrm{g\,cm^2},$$
$$I_m = 7{\cdot}2 \times 10^{44}\,\mathrm{g\,cm^2}.$$

The change in the length of the day is

$$\Delta T = -\frac{2\pi}{\omega_m^2}\Delta\omega_m = -\frac{2\pi}{\omega_m^2}\frac{I_c}{I}\Delta(\omega_m - \omega_2),$$

where $$I = I_c + I_m.$$

A decrease in the rate of westward drift by one-tenth of its steady-state value therefore lengthens the day by 1·3 ms.

Elsasser & Takeuchi showed that a reasonable change in the size of the toroidal field (a) at the core boundary was sufficient to provide a net couple on the mantle of the magnitude required to effect this transfer of angular momentum: in our notation their argument is that

$$I_m \Delta\omega_m/\Delta t = \tfrac{8}{15} g_1^0 d^3 \Delta M$$

so that if $\Delta T/\Delta t \sim 1{\cdot}3$ ms/decade, $\Delta M \sim 0{\cdot}05$ G, a fraction of its steady-state value.

Such arguments as these ignore the question of how rapidly the mantle responds to a change in the strength of the electromagnetic coupling at the core boundary. What happens when, through some disturbance of the velocity field in the outer core, the strength of the magnetic field just below the core boundary is changed? Obviously such a change results in a change in the size of the toroidal fields induced at the core boundary. As this change diffuses through the mantle, the mantle is subject to a change in the restoring forces provided by the toroidal-poloidal interactions. Consequently, it is also subject to a disturbance in angular velocity, consistent with the requirement that the angular momentum of core plus mantle remain constant. By Lenz's law this disturbance in ω_m is such as to reduce the production of excessive toroidal fields at the core boundary, and bring the mantle into equilibrium with the core once more. The problem of the mantle's response is a complicated one, as it combines in a non-linear way the mechanical problem of the mantle's motion under an applied couple changing with time and the electromagnetic problem of the diffusion of changes in field strength through an accelerated mantle.

Bullard *et al.* (1950, p. 90) quotes without derivation an expression for the time constant of the electromagnetic coupling between core and mantle, the time for ω_m to reach within e^{-1} of its new equilibrium value following a step-function disturbance in the coupling. In our notation, neglecting σ_1/σ with respect to 1,

$$\tau = \frac{15}{16}\frac{I_c I_m}{I}\frac{(g_1^0)^{-2}}{\pi\sigma_1 b^5}\left(\frac{b}{d}\right)^6 \frac{(1+\frac{3}{2}b^5/c^5)}{(1-b^5/c^5)}. \tag{25}$$

Bullard rejected the possibility of accounting for the observed irregularities in ω_m in the fashion just described, concluding that the coupling was not sufficiently tight for a reasonable change in field strength at the core boundary to register a change ΔT of a few milliseconds within a decade. The difficulty lay in his assumption that $\sigma_1 \sim 10^{-10}$ e.m.u., which gives $\tau \sim 500$ yr.

It was not until Runcorn (1955) considered the problem of the conductivity of the lower mantle in further detail that the difficulty was to some extent resolved. Runcorn pointed out that significant changes in the rate of secular variation establish themselves at the surface of the earth within at most a few years, and that this provides an argument for an upper limit to the conductivity of the lower 2000 km of the mantle, of about 10^{-9} e.m.u. With a higher conductivity the disturbances in field strength that take place at the core boundary would be masked for too long a time by the slow decay of the opposing eddy currents they induce in the lower mantle. This limit is below the upper limit set by a skin-depth argument similar to that carried out for the core. Runcorn's conclusions are supported by the detailed investigation of the distribution of electrical conductivity with depth in the mantle, carried out by McDonald (1957) and referred to in table 1. The data on the secular variation permit us therefore to increase by a factor of 10 the coupling provided by Bullard's mechanism, and so make it reasonable to ascribe the irregular changes in the length of the day to random fluctuations of the state of motion in the core.

Obviously the additional coupling provided by the rough, non-dipole magnetic topography at the core boundary will hasten the response of the mantle to changes in the magnetic-field régime in the outer core. We now investigate in rigorous fashion the mixed mechanical-electromagnetic problem posed above, using the coupling mechanism described in § 3. To render the problem linear, we consider only first-order perturbations from the steady-state angular velocity and field strength.

4·2. *Linearization of the mixed mechanical-magnetic problem*

Suppose that, beginning at time $t = 0$, the steady-state equilibrium described in § 3 is disturbed, owing to a change in the strength of the magnetic field penetrating the core boundary. During the disturbance the angular velocities of the mantle and outer core are $\omega_m^{(0)}+\omega_m^{(1)}$ and $\omega_2^{(0)}+\omega_2^{(1)}$, respectively. Here the superscripts (0) and (1) denote the steady-state values, and deviations from them, respectively. It is convenient, and the invariance of Maxwell's equations under uniform rotations allows us, to take our frame of reference as rotating uniformly with angular velocity $\omega_m^{(0)}$, as in § 3. An observer stationed in this frame finds a magnetic field in the mantle of magnitude

$$\mathbf{H} = \mathbf{H}^{(0)}+\mathbf{H}^{(1)},$$

where

$$\mathbf{H}^{(0)} = \sum_{n\geqslant 1}\sum_{m=0}^{n}(\mathbf{H}_{Pnm}^{(0)}+\mathbf{H}_{Tnm}^{(0)})\,\mathrm{e}^{-\mathrm{i}m\chi_0 t}$$

(in complex representation). Here the field present in the steady state has had its time factor separated for later convenience, and the extra field seen during the disturbance is $\mathbf{H}^{(1)}$.

The first-order perturbation equation resulting from (7) is

$$\frac{\partial \mathbf{H}^{(1)}}{\partial t} = -\frac{1}{4\pi\sigma_1}\operatorname{curl}\operatorname{curl}\mathbf{H}^{(1)}-\omega_m^{(1)}\sum_{k=1}^{3}\hat{e}_k\frac{\partial H_k^{(0)}}{\partial\phi}. \tag{26}$$

The last term on the right-hand side of (26) arises because the mantle sweeps past the steady-state field with an additional angular velocity $\omega_m^{(1)}$ during the perturbation, and so is subject to an extra electric current induced by the steady-state field.

The interactions among themselves of the steady-state fields contribute, of course, no net couple on the mantle. To the first order, therefore, the equation of motion of the mantle during the perturbation is

$$I_m\frac{\mathrm{d}\omega_m^{(1)}}{\mathrm{d}t} = -\frac{1}{4\pi}\int_{V_m} r\sin\theta\,\mathrm{d}V[\mathscr{R}(\mathbf{H}^{(0)})\times\mathscr{R}(\operatorname{curl}\mathbf{H}^{(1)})+\mathscr{R}(\mathbf{H}^{(1)})\times\mathscr{R}(\operatorname{curl}\mathbf{H}^{(0)})]_{\phi}. \tag{27}$$

We now suppose that the disturbance does not affect the dynamo action deep inside the core and write the conservation of angular momentum

$$I_m \omega_m^{(1)} + I_c \omega_2^{(1)} = 0. \tag{28}$$

In order to provide for disturbances of a delta- or step-function character, and to avoid the thorny questions of convergence, term-by-term differentiability, etc., that arise in a classical analysis of the time-dependent perturbation problem, we solve the problem using Fourier transforms and the concept of the 'generalized function' developed by Schwartz (for a description of the method see Lighthill 1958).

4·3. *Penetration of the mantle by disturbance field*

It is not difficult to show that (26) allows us to represent the first-order perturbation field in the mantle as the sum of poloidal and toroidal fields of different zonal and tesseral character:

$$\mathbf{H}^{(1)} = \sum_{n\geq 1} \sum_{m=0}^{n} (\mathbf{H}^{(1)}_{Pnm} + \mathbf{H}^{(1)}_{Tnm}),$$

where
$$\frac{\partial \mathbf{H}^{(1)}_{(\frac{P}{T})nm}}{\partial t} = -\frac{1}{4\pi\sigma_1} \operatorname{curl}\operatorname{curl} \mathbf{H}^{(1)}_{(\frac{P}{T})nm} - \mathrm{i}m\omega_m^{(1)} \mathbf{H}^{(0)}_{(\frac{P}{T})nm} \mathrm{e}^{-\mathrm{i}m\chi_0 t}. \tag{29}$$

We define the Fourier transforms

$$\mathscr{H}_{(\frac{P}{T})nm} = \int_{-\infty}^{\infty} \mathbf{H}^{(1)}_{(\frac{P}{T})nm} \mathrm{e}^{\mathrm{i}(p+m\chi_0)t}\, \mathrm{d}t,$$

$$\Omega(p) = \int_{-\infty}^{\infty} \omega_m^{(1)} \mathrm{e}^{\mathrm{i}pt}\, \mathrm{d}t.$$

Then if we write

$$\mathbf{X}_{(\frac{P}{T})nm} = \mathscr{H}_{(\frac{P}{T})nm} - \frac{m}{p}\Omega(p)\, \mathbf{H}^{(0)}_{(\frac{P}{T})nm},$$

$$k_1^2 = k^2 + 4\pi\sigma_1 \mathrm{i}p,$$

and make use of (8), the Fourier transform of the magnetic diffusion equation (29) reduces to

$$\operatorname{curl}\operatorname{curl} \mathbf{X} = k_1^2 \mathbf{X}. \tag{30}$$

We shall suppose the disturbance field at the surface of the Earth has a radial component

$$H^{(1)}_{r,Pnm}|_{r=d} = -(n+1)\,(b/d)^{n+2} F'^{(1)}_{nm}(t)\, P_{nm}\, \mathrm{e}^{\mathrm{i}m(\phi-\chi_0 t)}.$$

Physically, of course, any perturbation reflects a disturbance of the eddy structure in the upper core. Our simple model, which does not consider the magneto-hydrodynamics of the core, does not permit us to take this into account in any consistent way, i.e. to relate to one another the perturbations

$$\mathbf{H}^{(1)}_{Pnm}, \quad \mathbf{H}^{(1)(a)}_{T20}$$

just below the core boundary. Instead we simply assume that we are given the disturbance in the toroidal field of type (a) at the core boundary,

$$H^{(1)(a)}_{\phi,T20}|_{r=b} = -\tfrac{2}{3} M^{(1)}(t) \frac{\mathrm{d}P_{20}}{\mathrm{d}\theta} \tag{31}$$

and the first-order perturbation in each of the poloidal harmonics there, such that

$$H^{(1)}_{r,Pnm}|_{r=b} = -(n+1)\,[G_{Pnm}(t) - \mathrm{i}m\omega_2^{(1)} t F'_{nm}] P_{nm}\, \mathrm{e}^{\mathrm{i}m(\phi-\chi_0 t)}. \tag{32}$$

In (32) a distinction is made between the disturbance field due directly to changes in strength of the centres of non-dipole field—$G_{Pnm}(t)$—and the extra field described by the observer as the centres of non-dipole field move (attached to the perturbed outer layers of the core) with additional angular velocity $\omega_2^{(1)}$ past the reference frame.

(*a*) *Poloidal harmonics*

A comparison of (8) and (30) shows that in the conducting part of the mantle, $\mathbf{X}_{Pnm}$ is obtained from the exact expression for the time-independent part of the steady-state field

$$\mathbf{H}^{(0)}_{Pnm} = \mathbf{H}_{Pnm}\, e^{im\chi_0 t}|_{\text{steady}}$$

given in § 3·1, by replacing k and F'_{nm} by k_1 and $\mathscr{A}_{nm}(p)$ respectively. To satisfy (32) we require

$$\mathscr{A}_{nm}(p) = \mathscr{Q}_{nm}(p)\left[\mathscr{G}_{Pnm}(p) + im\frac{I_m}{I_c}\alpha(p)\,F'_{nm} - \frac{m}{p}\Omega(p)\,F'_{nm}\right],$$

where we have defined the Fourier transforms

$$\mathscr{G}_{Pnm}(p) = \int_{-\infty}^{\infty} G_{Pnm}(t)\, e^{ipt}\, dt,$$

$$\alpha(p) = \int_{-\infty}^{\infty} \omega_m^{(1)}(t)\, t\, e^{ipt}\, dt,$$

and the quantity $\quad [\mathscr{Q}_{nm}(p)]^{-1} = \tfrac{1}{2}\pi(b/c)^{n+2}\,(k_1 c)^{\frac{5}{2}} Y_{n-\frac{1}{2}}(k_1 c)\mathscr{N}_n^{n-1}(k_1 b).$

Continuity at $r = c$ is ensured by having

$$f_{nm}(p) = \int_{-\infty}^{\infty} F'^{(1)}_{nm}(t)\, e^{ipt}\, dt = \mathscr{A}_{nm}(p) + \frac{m}{p}\Omega(p)\,F'_{nm}.$$

Taking the inverse transform of this last equation gives

$$\begin{aligned} F'^{(1)}_{nm}(t) = \frac{1}{2\pi}\int_{-\infty}^{t} G_{Pnm}(\lambda)\, d\lambda \int_{-\infty}^{\infty} \mathscr{Q}_{nm}\, e^{-ip(t-\lambda)}\, dp \\ + \frac{im}{2\pi}\frac{I_m}{I_c} F'_{nm}\int_{-\infty}^{t} \omega_m^{(1)}(\lambda)\,\lambda\, d\lambda \int_{-\infty}^{\infty} \mathscr{Q}_{nm}\, e^{-ip(t-\lambda)}\, dp \\ + \frac{mF'_{nm}}{2\pi}\int_{-\infty}^{t} \omega_m^{(1)}(\lambda)\, d\lambda \int_{-\infty}^{\infty} (1-\mathscr{Q}_{nm})\,\frac{e^{-ip(t-\lambda)}}{p}\, dp. \end{aligned} \tag{33}$$

Once the operations requiring the use of generalized function theory are completed, we cut off the doubly-infinite integration over the variable time parameter λ at $\lambda = t$, since 'future' perturbations have no physical effect on $F'^{(1)}_{nm}(t)$. The three terms of (33) are respectively due to:

(1) the penetration into the mantle of a change in the strength of the magnetic field in the core;

(2) the creation (in the observer's frame of reference) of extra field due to the additional rotation $\omega_2^{(1)}$ given the non-dipole field centres;

(3) the screening of steady-state fields due to the extra angular velocity $\omega_m^{(1)}$ of the mantle past these fields.

As soon as a perturbation in field strength is introduced at the core-mantle boundary, induced currents flow in the bottom of the mantle in such a way as to oppose the penetration of the mantle by the excited field. These eddy currents decay in a time proportional to the square of the dimensions of the excitation, i.e. to the square of the 'wavelength' of the excited mode. The higher the harmonics excited, the more rapid the decay of these eddy currents and the faster the excitation penetrates into the bottom of the mantle. The same thing happens all the way up through the conducting part of the mantle; a disturbance at $r = c$ registers immediately at the top of the mantle. Consequently an excitation in the higher harmonics registers at $r = d$ much more quickly than one in the lower harmonics.

Runcorn (1955) refers to observations on sudden changes in the rate of secular variation, which become effective within about 5 years. Clearly these phenomena are described, in our model, by (33). It can be shown, using the results of § 4·5, that the second and third terms of (33) are small compared to the first, for reasonable perturbations in field strength and durations in time of a few decades. The time required for penetration of the mantle by disturbance field is clearly determined by the poles of $\mathscr{Q}_{nm}$ at

$$p_{nm,\,j} = -m\chi_0 - \mathrm{i}(k_{1,\,nm,\,j})^2/4\pi\sigma_1,$$

where the roots of

$$J_{n+\frac{1}{2}}(k_1 b)\,Y_{n-\frac{1}{2}}(k_1 c) - J_{n-\frac{1}{2}}(k_1 c)\,Y_{n+\frac{1}{2}}(k_1 b) = 0$$

form a real sequence $k_{1,\,nm,\,j}$ ($j = 1, 2, 3, \ldots$). Extending the contour of integration by a semi-circular arc of infinite radius in the lower half of the complex p-plane, we have

$$F'^{(1)}_{nm}(t) \sim -\mathrm{i}\int_{-\infty}^{t} \mathscr{G}_{Pnm}(\lambda)\,\mathrm{e}^{\mathrm{i}m\chi_0(t-\lambda)}\,\mathrm{d}\lambda \times \sum_{j\geq 1}(\text{residue of } \mathscr{Q}_{nm} \text{ at } p_{nm,\,j})\,\mathrm{e}^{-(t-\lambda)/\tau_{nm,\,j}},$$

where we define

$$\tau_{nm,\,j} = \frac{4\pi\sigma_1}{(k_{1,\,nm,\,j})^2}.$$

It takes a time of order $\tau_{nm,1}$ for a step-function disturbance, applied at the core boundary at $t = 0$, to register appreciably at the top of the mantle. An upper limit to this 'registry time' is given by $\tau_{10,\,1}$. With the geometry of our model of the mantle,

$$k_{1,10,1} = 0{\cdot}94 \times 10^{-8}\,\mathrm{cm}^{-1}.$$

If $\tau_{10,1} \leq 5$ yr, then $\sigma_1 \leq 10^{-9}$ e.m.u. This argument is no different in principle from that given by Runcorn (1955), who considered the penetration of a semi-infinite mantle by a step-function disturbance applied uniformly at the plane boundary.

(b) *Toroidal harmonics generated at the core boundary*

Invoking (3) and (28), and neglecting σ_1/σ with respect to 1, we have

(i) $m = 0$

$$\sum_N \frac{\partial}{\partial r}[rH^{(1)}{}^{Pn0}_{\phi,\,TN0}]_{r=b+} = 4\pi\sigma_1\chi_0 b^2 \sin\theta\left[H^{(1)}_{r,\,Pn0} - \frac{I}{I_c}\frac{\omega^{(1)}_m}{\chi_0}H^{(0)}_{r,Pn0}\right]_{r=b},$$

(ii) $m \neq 0$

$$\sum_N N(N+1)\frac{\partial}{\partial r}[rH^{(1)}{}^{Pnm}_{\theta,\,TNm}]_{r=b+} = \frac{(kb)^2}{\sin^2\theta}\frac{\partial}{\partial\theta}\left\{\sin^2\theta\left[H^{(1)}_{r,Pnm} - \frac{I}{I_c}\frac{\omega^{(1)}_m}{\chi_0}H^{(0)}_{r,Pnm}\,\mathrm{e}^{-\mathrm{i}m\chi_0 t}\right]_{r=b}\right\},$$

where $N = n \pm 1$. Comparison with § 3·1 shows that the transform field $\mathbf{X}^{Pnm}_{TNm}$ is obtained from the expressions (11) to (14) for

$$\mathbf{H}^{(0)}{}^{Pnm}_{TNm} = \mathbf{H}^{Pnm}_{TNm}\,\mathrm{e}^{\mathrm{i}m\chi_0 t}|_{\text{steady}}$$

by replacing k, C^{Pnm}_{TNm}, K_N and F'_{nm} by k_1, $B^{Pnm}_{TNm}(p)$,

$$\mathscr{K}_N(p) = k_1 b \mathscr{N}^N_{N-1}(k_1 b)/\mathscr{N}^N_N(k_1 b),$$

and
$$\mathscr{G}_{Pnm}(p) + \mathrm{i}m\frac{I_m}{I_c}\alpha(p)\,F'_{nm} - \left(\frac{I}{I_c}+\frac{m\chi_0}{p}\right)\frac{\Omega(p)}{\chi_0}\,F'_{nm},$$

respectively. Numerical calculations, using the value of $p = \bar{p}$ obtained from § 4·4 show that for all m and for $N > 1$ we can neglect $|\mathscr{K}_N(\bar{p})|$ with respect to N, and that

$$\mathscr{K}_1(\bar{p})|_{m=0} \sim \mathscr{K}_1(\bar{p})|_{m=1} \sim -\tfrac{2}{3}.$$

($|\mathscr{K}_N(\bar{p})|$ is smaller than $|K_N|$ because the boundary condition at $r = c$ is relatively less important for diffusing than for steady-state fields, where its effect has had time to fully influence the distribution of field strength below $r = c$.)

(*c*) *Perturbations in toroidal field* (*a*)

The disturbance (31) diffuses through the lower mantle without screening, and the transform field

$$\mathscr{H}_{\phi,\,T20}^{(a)} = -\tfrac{2}{3}\mathscr{M}(p)\,\frac{r}{b}\,\frac{\mathscr{N}_2^2(k_1 r)}{\mathscr{N}_2^2(k_1 b)}\,\frac{\mathrm{d}P_{20}}{\mathrm{d}\theta},$$

where
$$\mathscr{M}(p) = \int_{-\infty}^{\infty} M^{(1)}(t)\,\mathrm{e}^{\mathrm{i}pt}\,\mathrm{d}t.$$

4·4. *Time constant of the mantle-core coupling*

Taking the Fourier transform of (27) we have

$$-\mathrm{i}pI_m\Omega(p) = \sum_{k=1}^{6}\mathscr{C}_k, \tag{34}$$

where

$$\mathscr{C}_1 = -\frac{1}{16\pi}\sum_{n\geqslant 3}\sum_{m=0}^{n-2}(1+\delta_{m0})\int_{S_c} r\sin\theta\,\mathrm{d}S\,[H^{(0)}_{r,Pnm}\,\mathscr{H}^{P,n-2,m*}_{\phi,T,n-1,m} + \mathscr{H}^*_{r,Pnm}\,H^{(0)P,n-2,m}_{\phi,T,n-1,m} + \text{c.c.}]_{r=b}$$

and $\mathscr{C}_2$ to $\mathscr{C}_4$ are obtained in an analogous fashion indicated in § 3·2,

$$\mathscr{C}_5 = -\frac{1}{16\pi}\sum_{n\geqslant 1}\sum_{m=1}^{n}\int_{S_c} r\sin\theta\,\mathrm{d}S\,[H^{(0)}_{r,Pnm}\,\mathscr{H}^*_{\phi,Pnm} + \mathscr{H}^*_{r,Pnm}\,H^{(0)}_{\phi,Pnm} + \text{c.c.}]_{r=b},$$

$$\mathscr{C}_6 = -\frac{1}{4\pi}\sum_{n=1,3}\int_{S_c} r\sin\theta\,\mathrm{d}S\,[H^{(0)}_{r,Pn0}\,\mathscr{H}_{\phi,\,T20}^{(a)} + \mathscr{H}_{r,\,Pn0}\,H^{(0)\,(a)}_{\phi,\,T20}]_{r=b}.$$

We note that the complex conjugate of a Fourier-transformed quantity is obtained by replacing ($\mathrm{i}m, \mathrm{i}p$) by ($-\mathrm{i}m, \mathrm{i}p$).

Using the results of § 4·3 we find that

$$\mathscr{C}_1 = \sum_{n\geqslant 3}\sum_{m=0}^{n-2}\Gamma_{Pnm;\,T,n-1,m}^{\;P,n-2,m}\left[\mathscr{R}\left(\frac{F'_{nm}F'^*_{n-2,m}}{n-1-K^*_{n-1}}\right)\right]^{-1}$$
$$\times\left\{\mathscr{R}\left(\frac{\mathscr{G}_{Pnm}(p)\,F'^*_{n-2,m}}{n-1-K^*_{n-1}}\right) + \mathscr{R}\left(\frac{F'_{nm}\mathscr{G}^*_{P,n-2,m}(p)}{n-1-\mathscr{K}_{n-1}(p)}\right) - \frac{I}{I_c}\frac{\Omega(p)}{\chi_0}\,\mathscr{R}\left(\frac{F'_{nm}F'^*_{n-2,m}}{n-1-\mathscr{K}^*_{n-1}(p)}\right)\right\}$$

and $\mathscr{C}_2$–$\mathscr{C}_4$ can be written down immediately by inspection. We have ignored the contributions to $\mathscr{C}_1$–$\mathscr{C}_4$ due to the ponderomotive interactions with steady-state fields of:

(1) the extra field created in the observer's frame by the perturbation $\omega_2^{(1)}$ given the rotation of the field sources in the upper core;

(2) the shielding fields induced from steady-state fields by the additional angular velocity $\omega_m^{(1)}$ of the mantle. Neglect of these effects removes from the summands of $\mathscr{C}_1$–$\mathscr{C}_4$ terms proportional to

$$\left(\frac{I_m}{I_c}\alpha(p), \frac{\Omega(p)}{p}\right) m\Gamma_{Pnm;\,{Pvm \atop TNm}}$$

Such terms can be shown, by a numerical calculation using the $p = \bar{p}$ obtained at the end of this section by ignoring them, to contribute negligibly to the tightness of the mantle-core coupling in comparison with the terms proportional to

$$\frac{I}{I_c}\frac{\Omega(p)}{\chi_0}\Gamma_{Pnm;\,{Pvm \atop TNm}}.$$

which we have retained.

To the same approximation

$$\mathscr{C}_5 = \sum_{n\geq 1}\sum_{m=1}^{n}\Gamma_{Pnm;\,Pnm}\left\{\mathscr{R}\left(\frac{\mathscr{G}_{Pnm}(p)}{F'_{nm}}\right) + \frac{2n-1}{4\pi\sigma_1 m\chi_0 b^2}\left[1-\left(\frac{b}{c}\right)^{2n-1}\right]^{-1}\mathscr{I}\left(\frac{\mathscr{L}_n(p)\mathscr{G}_{Pnm}(p)}{F'_{nm}}\right)\right\},$$

where

$$\mathscr{L}_n(p) = \frac{k_1 b\mathscr{N}_{n-1}^{n-1}(k_1 b)}{\mathscr{N}_n^{n-1}(k_1 b)}.$$

Also

$$\mathscr{C}_6 = -\frac{b^3}{3}\sum_{n=1,3}(n+1)I_{n0,\,20}[\mathscr{M}(p)F'_{n0} + M\mathscr{G}_{Pn0}(p)],$$

where $I_{n0,\,20}$ is given by (17).

The inverse transform of (34) yields the following expression for the perturbation in angular velocity of the mantle as a function of time

$$\omega_m^{(1)}(t) = \frac{1}{2\pi}\int_{-\infty}^{t} d\lambda\int_{-\infty}^{\infty}\frac{\mathscr{E}(\lambda,p)\,e^{-ip(t-\lambda)}}{(IW/I_c\chi_0 - ipI_m)}\,dp, \tag{35}$$

where the integration over λ is again cut off at $\lambda = t$ to obtain a physically meaningful result, once inversion is completed. We use $\mathscr{E}(\lambda,p)$ to designate the sum of the expressions obtained from

(1) to (4) $\mathscr{C}_1$–$\mathscr{C}_4$ by removing the terms proportional to

$$\frac{I}{I_c}\frac{\Omega(p)}{\chi_0}\Gamma_{Pnm;\,{Pvm \atop TNm}}$$

and replacing $\mathscr{G}_{P\nu m}(p)$ by $G_{P\nu m}(\lambda)$ throughout;

(5) $\mathscr{C}_5$ by replacing $\mathscr{G}_{P\nu m}(p)$ by $G_{P\nu m}(\lambda)$ throughout;

(6) $\mathscr{C}_6$ by replacing $\mathscr{M}(p)$ and $\mathscr{G}_{P\nu 0}(p)$ by $\mathscr{M}^{(1)}(\lambda)$ and $G_{P\nu 0}(\lambda)$, respectively.

Here

$$W = \pi\sigma_1\chi_0 b^5\sum_{k=1}^{4}S_k, \tag{36}$$

where the S_k are the sums (uncorrected for K_N) given in § 3·2. The sums S_k, cut off at $n = 6$, are found using the analysis of the earth's field due to Finch & Leaton (1957)

$$S_1 = S_4 = 0{\cdot}38\text{ e.m.u.},\quad S_2 = 0{\cdot}66\text{ e.m.u.},\quad S_3 = 4{\cdot}56\text{ e.m.u.}$$

In writing (36) we have ignored $\mathscr{K}_N(p)$—the effect of taking it into account is to reduce the tightness of the mantle-core coupling by 1%.

Qualitatively it is clear that a decrease, say, in the poloidal field strength weakens the retarding couple on the mantle, so that the mantle must accelerate to a new equilibrium angular velocity. The rate of this acceleration is governed by the rate of change of field strength appearing explicitly in the expression $\mathscr{E}(\lambda, p)$, and by the tightness of the electromagnetic coupling of the mantle to the core, which we now investigate.

To evaluate

$$\int_{-\infty}^{\infty} \frac{\mathscr{E}(\lambda, p)\, e^{-ip(t-\lambda)}}{(IW/I_c\chi_0 - ipI_m)}\, dp \tag{37}$$

we complete the path of integration by a semi-circular arc of infinite radius drawn in the lower half of the complex p-plane, and use the theory of residues as in (33). Physically this procedure is indicated by the stability of the equilibrium between core and mantle, which requires that the poles of the integrand of (37) be at points p such that $e^{-ip(t-\lambda)}$ decays in time. Insofar as W (corrected for $\mathscr{K}_N(p)$) is a function of p, the required poles are many, located at the roots of

$$IW/I_c\chi_0 - ipI_m = 0. \tag{38}$$

However, the roots of (38) are so tightly clustered about

$$\bar{p} = -i\frac{I}{I_c I_m}\pi\sigma_1 b^5 \sum_{k=1}^{4} S_k \tag{39}$$

that with an error of less than 1% we may regard all the poles as coalescing into one at $p = \bar{p}$. Then (37) reduces to

$$(2\pi/I_m)\,\mathscr{E}(\lambda, \bar{p})\, e^{-(t-\lambda)/\tau},$$

where

$$\tau = \frac{I}{i\bar{p}} = \frac{I_c I_m}{I}\left(\pi\sigma_1 b^5 \sum_{k=1}^{4} S_k\right)^{-1} \tag{40}$$

is the time constant of the mantle-core coupling.

If we take $\mathscr{K}_N(p)$ into account, the terms $S_{2,20}$ and $S_{2,21}$ are multiplied by 0·60 and other terms are only negligibly affected. The resulting value of τ is increased by 1% over the value given by (40). At this point we may also return to examine, and verify using $\bar{p}$, the approximations made earlier in this section and in §4·3 (*b*).

Physically speaking, we would expect the time constant of the coupling to have different values according as the coupling was effected by the presence of a poloidal dipole, quadrupole, ...field at the core boundary. In the presence of a field made up of all these harmonics we would then expect to find the mantle's response to a step-function disturbance in the coupling to be a complicated function of time, taking into acount all the different time constants provided by the different independent couplings. The substance of our investigations is that the actual response of the mantle to such a disturbance is well characterized by a single 'smeared-out' time constant given by (40).

Obviously the tightness of the coupling is very sensitive to the value assumed for σ_1. Setting $\sigma_1 \sim 10^{-9}$ e.m.u. we find

$$\tau \sim 25\,\text{yr}.$$

This result will not be much reduced by taking into account the remainder of the earth's field (the harmonics $n > 6$).

If the earth's magnetic field is approximated by the main dipole, (40) reduces to

$$\tau_B = \frac{I_c I_m}{I}(\pi\sigma_1 b^5 S_{3,10})^{-1} = \frac{15}{16}\frac{I_c I_m}{I}\frac{(g_1^0)^{-2}}{\pi\sigma_1 b^5}\left(\frac{b}{d}\right)^6. \tag{41}$$

The discrepancy between (25) and (41) is probably due to Bullard's approximate treatment of the problem, neglecting magnetic diffusion, in which $K_2|_{m=0}$, rather than $\mathscr{K}_2(\bar{p}_B)|_{m=0}$, appears. For $\sigma_1 \sim 10^{-9}$ e.m.u.,

$$\tau_B \sim 40\,\text{yr}.$$

The mechanical effects of the non-dipole part of the earth's field are strong enough to increase the tightness of the electromagnetic coupling of the mantle to the core by 60% over that afforded by the coupling mechanism considered by Bullard. Moreover, the tightness of the coupling depends overwhelmingly on the attempt of the mantle, in obedience to Lenz's law, to reduce the production of excessive toroidal field at the core boundary, and hardly at all on its attempt to overcome the screening of the fields produced in the upper core and taking part in the westward drift.

4·5. *Irregularities in the length of the day*

The change in the length of the day, as a function of time, is obtained from (35)

$$\Delta T(t) = -\frac{2\pi}{I_m\omega_m^2}\int_{-\infty}^{t}\mathscr{E}(\lambda,\bar{p})\,\mathrm{e}^{-(t-\lambda)/\tau}\mathrm{d}\lambda.$$

The complete formal expression for $\mathscr{E}(\lambda,\bar{p})$, defined in § 4·4, is in general very complicated. It may be somewhat simplified by noting that when K_N and $\mathscr{K}_N(\bar{p})$ are comparable with N, they are real or have only a negligible imaginary part, and that to a good approximation

$$\mathscr{I}(\mathscr{L}_n(\bar{p})) \sim \frac{4\pi\sigma_1 m\chi_0 b^2}{2n-1}\left\{1-\left(\frac{b}{c}\right)^{2n-1}\right\}.$$

Then

$$\frac{\mathscr{E}(\lambda,\bar{p})}{\pi\sigma_1\chi_0 b^5} \sim 2\sum_{n\geqslant 1}\sum_{m=0}^{n} S_{4,nm}^{(c)}\frac{\mathscr{R}[G_{Pnm}(\lambda)\,F'^{*}_{n+2,m}]}{\mathscr{R}[F'_{nm}F'^{*}_{n+2,m}]} + 2\sum_{n\geqslant 1}\sum_{m=0}^{n} S_{4,nm}\frac{\mathscr{R}[F'_{nm}G^{*}_{P,n+2,m}(\lambda)]}{\mathscr{R}[F'_{nm}F'^{*}_{n+2,m}]}$$
$$+\sum_{n\geqslant 2}\sum_{m=0}^{n-1}[S_{2,nm}^{(c)}+S_{2,nm}]\frac{\mathscr{R}[G_{Pnm}(\lambda)\,F'^{*}_{nm}]}{\mathscr{R}[F'_{nm}F'^{*}_{nm}]} + \sum_{n\geqslant 1}\sum_{m=0}^{n}[S_{3,nm}^{(c)}+S_{3,nm}]\frac{\mathscr{R}[G_{Pnm}(\lambda)\,F'^{*}_{nm}]}{\mathscr{R}[F'_{nm}F'^{*}_{nm}]}$$
$$+2\sum_{n\geqslant 1}\sum_{m=1}^{n} S_{5,nm}\,\mathscr{R}\left[\frac{G_{Pnm}(\lambda)}{F'_{nm}}\right] - \frac{(n+1)\,(\delta_{n1}+\delta n_3)}{3\pi\sigma_1\chi_0 b^2}[MG_{Pn0}(\lambda)+F'_{n0}M^{(1)}(\lambda)].$$

We shall in this paper restrict our attention to (i) step-function disturbances, (ii) linear changes with time in field strength, applied at the core boundary beginning at $t=0$. For simplicity, suppose the disturbances are such that

(i) $G_{Pnm}(t) = 0 \quad (n=1),$
$\qquad = xF'_{nm} \quad (n>1),$
$\qquad M^{(1)}(t) = yM,$

(ii) $G_{Pnm}(t) = 0 \quad (n=1),$
$\qquad = (xt/\tau)\,F'_{nm} \quad (n>1),$
$\qquad M^{(1)}(t) = (yt/\tau)\,M,$

where x and y are real fractions. Then after some algebra it can be shown that in both cases

$$\mathscr{E}(\tau,\bar{p}) = \pi\sigma_1\chi_0 b^5(3{\cdot}51x-5{\cdot}40y).$$

In case (i)

$$\Delta T(t) = -\frac{2\pi\tau}{I_m\omega_m^2}\mathscr{E}(\tau,\bar{p})\,(1-\mathrm{e}^{-t/\tau}) \tag{42}$$

and in case (ii)

$$\Delta T(t) = -\frac{2\pi\tau}{I_m\omega_m^2}\mathscr{E}(\tau,\bar{p})\left[\frac{t}{\tau}-1+\mathrm{e}^{-t/\tau}\right]. \tag{43}$$

A comparison of harmonic analyses of the geomagnetic field made at 10 yr intervals (Vestine, Lange, Laporte & Scott 1947; Finch & Leaton 1957) establishes that the assumption made for the steady-state calculations of § 3·1, namely $(g_n^m)^2+(h_n^m)^2=$ constant for each zonal and tesseral harmonic, is quite good. In a decade the change in strength of most of the non-dipole harmonics is of the order of 10%, or less. This sets an upper limit to the factor x, namely $x = 0{\cdot}1$ in case (i) and $x = 0{\cdot}25$ in case (ii). It seems unlikely that the amount of toroidal field (a) reaching the core boundary could change by much more than 10% in a decade, since this change would result from whatever changes in the velocity field affect the non-dipole poloidal field. For the greatest effect, $y = -x$. Disturbances of this magnitude are consistent with the restriction to first-order perturbation theory within times of the order of a decade or two following onset.

With these values for the disturbance field,

$$\Delta T\,(10\,\text{yr}) \sim 0{\cdot}6\,\text{ms} \quad \text{in case (i)},$$
$$\sim 0{\cdot}3\,\text{ms} \quad \text{in case (ii)}.$$

A mean electrical conductivity in the lower mantle, $\sigma_1 \sim 2{\cdot}5 \times 10^{-9}$ e.m.u. is within the limits of error of McDonald's (1957) estimate. This would reduce the time constant of the mantle-core coupling to 10 yr. For perturbations $x = -y = 0{\cdot}1$,

$$\Delta T\,(10\,\text{yr}) \sim 1{\cdot}2\,\text{ms} \quad \text{in case (i)},$$
$$\sim 0{\cdot}7\,\text{ms} \quad \text{in case (ii)}.$$

These simple examples lead us to conclude that the transfer of angular momentum between core and mantle, electromagnetically coupled by the mechanism here discussed, can indeed be invoked to explain the irregular changes in the length of day taking place at the rate of a millisecond per decade. A more rapid rate of change in the period of the earth's rotation, of order 0·4 ms/yr, has recently been observed (Essen, Parry, Markowitz & Hall 1958). To account for this rate of change using the above models we require a much tighter coupling, with $\sigma_1 \sim 6\text{–}10 \times 10^{-9}$ e.m.u. This seems excessive in the light of the data of table 1, and would gravely attenuate the secular variation. It thus appears difficult to ascribe so rapid a rate of change in the length of the day to a change in the coupling of the mantle to the core.

In the absence of any other likely geophysical cause for the irregular millisecond/decade rates of change in the length of day, the arguments here presented may be reversed to give a lower limit of about 10^{-9} e.m.u. to the mean electrical conductivity in the bottom 2000 km of the mantle.

5. Conclusions

In this paper the intimate connexion between motions in the earth's fluid core, producing and modifying the geomagnetic field, and the rate of rotation of the electromagnetically-coupled mantle has been examined with the aid of a model resting on several simplifying assumptions. For instance, the large toroidal field generated in the deep interior of the core may not reach the core boundary with anything like simple quadrupole symmetry. The eddy régime in the core which would distort this toroidal field is also responsible for the spatial production of non-dipole field and the consequent westward drift of different harmonics at different rates, and at rates which change with time. To attempt to take these different rates into account, and to discuss the transfer of angular momentum to a convecting, non-uniformly rotating core, would enormously increase the difficulties of the analysis and would be superfluous without more detailed knowledge of the magneto-hydrodynamics of the core.

Runcorn (personal communication) has suggested that the westward drift is really a transient phenomenon. From §§2 and 4 of this paper it is clear that as long as the present polarity and strength of the main dipole are maintained by a dynamo process of the kind envisaged by Elsasser, Bullard and Parker, there must be a westward drift of the upper layers of the core past the mantle. If there is not, an electromagnetic couple of some 10^{25} dyn cm will act to bring about a westward drift within a few decades.

Vestine (1953) examined the rate of westward drift of one feature, the pole of the eccentric dipole field, and found it to change by as much as 30% in a decade, while the length of day changed by almost 2 ms. This is about twice as great a change in the rate of westward drift as the simple model of §4·1 would predict for the same change in the length of day. The discrepancy could be removed by abandoning the rigid-sphere model for the core and supposing that only the upper layers (say 500 km) of the core exchange angular momentum with the mantle, while the deep interior remains relatively undisturbed in its motion. This would result in a change in the strength of toroidal field (a) generated within the core, which we must assume could be sustained by the dynamo. The reduction in I_c would also tighten the coupling of the mantle to the upper core by a factor of two, but this would not make easier the explanation of the recently observed rapid rates of change in the length of day. The reductions in I_c and τ cancel one another in the expressions (42) and (43).

With this reduction in I_c both the observed and theoretically predicted changes in the rate of westward drift are not greater than 30% of the mean rate in times of the order of τ. Provided we restrict ourselves to times within an interval of this duration it should be satisfactory to assume, as in the present model, a westward drift subject to small perturbations from an equilibrium rate equal to the mean rate of drift during that time. It is to be hoped that determinations of the rate of westward drift of the non-dipole field at decade intervals will become available in the future.

In §4·3b we found that the diffusion into the mantle of toroidal field produced at the core boundary is not much affected by the boundary condition at $r = c$, in other words by the distribution of electrical conductivity far above the bottom of the mantle. This is reflected in the insensitivity of τ to $\mathscr{K}_N(\bar{p})$. The tightness of the coupling, given by (40), is then determined primarily by the electrical conductivity in a relatively small thickness of the mantle just above the core boundary. McDonald's best estimate of the conductivity of the mantle at the core boundary is $2{\cdot}2 \times 10^{-9}$ e.m.u., but it may be higher by a factor of two or three (McDonald 1957). This would enable us to attribute the very rapid rates of change in the rate of the earth's rotation, discussed in §4·5, to perturbations in the electromagnetic coupling of the mantle to the core. It should not be difficult to extend the present model to take into account the variation of conductivity with depth in the lower mantle, so as to test this conclusion.

This paper constitutes a Ph.D. dissertation for the University of Utah. Most of the work was done at the Scripps Institution of Oceanography, La Jolla, California. The author wishes to thank Professor W. M. Elsasser for his initial suggestion of the problem, his constant encouragement and helpful criticism. His thanks are due particularly to the National Research Council of Canada for a Special Scholarship, and to the Office of Naval Research and the Institute of Geophysics of the University of California, for additional support.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

References

Brouwer, D. 1952 *Proc. Nat. Acad. Sci., Wash.* **38**, 1.
Bullard, E. C. 1949*a* *Proc. Roy. Soc.* A, **197**, 433.
Bullard, E. C. 1949*b* *Proc. Roy. Soc.* A, **199**, 413.
Bullard, E. C., Freedman, C., Gellman, H. & Nixon, J. 1950 *Phil. Trans.* A, **243**, 67.
Elsasser, W. M. 1950 *Rev. Mod. Phys.* **22**, 1.
Elsasser, W. M. 1956 *Rev. Mod. Phys.* **28**, 135.
Elsasser, W. M. & Munk, W. 1958 *Contributions in geophysics: In honor of Beno Gutenberg* (ed. Benioff *et al.*), p. 228. New York: Pergamon Press.
Elsasser, W. M. & Takeuchi, H. 1955 *Trans. Amer. Geophys. Un.* **36**, 584.
Essen, L., Parry, J. V. L., Markowitz, W. & Hall, R. G. 1958 *Nature, Lond.* **181**, 1054.
Finch, H. F. & Leaton, B. R. 1957 *Mon. Not. R. Astr. Soc., Geophys. Suppl.* **7**, 314.
Lahiri, B. N. & Price, A. T. 1939 *Phil. Trans.* A, **237**, 509.
Lighthill, M. J. 1958 *An introduction to Fourier analysis and generalized functions.* Cambridge University Press.
Lowes, F. J. 1955 *Ann. Geophys.* **11**, 91.
McDonald, K. L. 1955 *J. Geophys. Res.* **60**, 377.
McDonald, K. L. 1957 *J. Geophys. Res.* **62**, 117.
Munk, W. & Revelle, R. 1952 *Mon. Not. R. Astr. Soc. Geophys. Suppl.* **6**, 331.
Rikitake, T. 1952 *Bull. Earthq. Res. Inst. Tokyo*, **30**, 191.
Rikitake, T. 1956 *Bull. Earthq. Res. Inst. Tokyo*, **34**, 283.
Runcorn, S. K. 1955 *Trans. Amer. Geophys. Un.* **36**, 191.
Stratton, J. A. 1941 *Electromagnetic theory.* New York: McGraw-Hill Book Company.
Tamao, T. 1959 *Sci. Rep. Tohoku Univ.*, 5th series, Geophysics **10**, 137.
Vestine, E. H. 1953 *J. Geophys. Res.* **58**, 127.
Vestine, E. H., Lange, I., Laporte, L. & Scott, W. E. 1947 *The geomagnetic field, its description and analysis*, publication 580, Carnegie Institute of Washington.
Watson, G. N. 1944 *Theory of Bessel functions* (2nd ed.). Cambridge University Press.
Whitham, K. 1958 *Canad. J. Phys.* **36**, 1372.

MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES
PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A